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DETERMINANTS CONNECTED WITH HOMOGENEOUS PRODUCTS AND SYMMETRIC FUNCTIONS OF THE ROOTS OF AN EQUATION

BY

K. BALACHANDRAN, *University of Travancore.*

1. **Notation** :—$\alpha, \beta, \gamma, \ldots \mu$ being any n numbers, we write

$\sum(n, r) = \alpha^r + \beta^r + \gamma^r + \cdots + \mu^r.$

$H(n, r)$ = sum of homogeneous products of r dimensions formed out of these n numbers, repetition being allowed.

$S(n, r)$ = sum of the products formed r at a time out of these n numbers.

When there is no ambiguity we shall write $\sum_r$, H_r and S_r for these. Also

$$\Delta(n, r) = \begin{vmatrix} \sum_1 & 1 & 0 & 0 & \cdot & \cdot & 0 \\ \sum_2 & \sum_1 & 2 & 0 & \cdot & \cdot & 0 \\ \sum_3 & \sum_2 & \sum_1 & 3 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sum_r & \sum_{r-1} & \sum_{r-2} & \sum_{r-3} & \cdot & \cdot & \sum_1 \end{vmatrix}$$

2. **Expressions for sums of homogeneous products.**

(*a*) *Expression for $H(n, r)$*

Let $F(x) = (1-\alpha x)^{-1}(1-\beta x)^{-1} \cdots (1-\mu x)^{-1}$ ... (1)

Then, if $\alpha x, \beta x, \ldots \mu x$ are all numerically less than unity, we have

$$F(x) = (1 + \alpha x + \alpha^2 x^2 + \cdots)(1 + \beta x + \beta^2 x^2 + \cdots) \cdots (1 + \mu x + \mu^2 x^2 + \cdots)$$
$$= 1 + H(n, 1)x + H(n, 2)x^2 + \cdots + H(n, r)x^r + \cdots \quad (2)$$

Also from (1),

$$\log F(x) = -[\log(1-\alpha x) + \log(1-\beta x) + \cdots + \log(1-\mu x)]$$
$$= \left(\sum_1 x + \sum_2 \frac{x^2}{2} + \sum_3 \frac{x^3}{3} + \cdots\cdots\right) \quad \ldots (3)$$

Differentiating (3) we have

$$F'(x) = F(x)[\sum_1 + \sum_2 x + \sum_3 x^3 + \cdots\cdots]$$

and substituting in this for $F(x)$ and $F'(x)$ from (2),

$$H(n,1)+2H(n,2)x+\cdots+rH(n,r)x^{r-1}+\cdots$$
$$=[1+H(n,1)x+H(n,2)x^2+\cdots][\Sigma_1+\Sigma_2x+\Sigma_3x^2+\cdots] \qquad (4)$$

Equating coefficients of like powers on either side of (5), we have

$$\left.\begin{array}{ll}\Sigma_1-H_1 & =0\\ \Sigma_2+H_1\Sigma_1-2H_2 & =0\\ \Sigma_3+H_1\Sigma_2+H_2\Sigma_1-3H_3 & =0\\ \cdots\cdots & \\ \Sigma_r+H_1\Sigma_{r-1}+H_2\Sigma_{r-2}+\cdots+H_{r-1}\Sigma_1-rH_r & =0\end{array}\right\} \qquad \ldots (5)$$

From these r equations eliminating the $(r-1)$ quantities $H_1, H_2, \ldots H_{r-1}$ we get at once

$$H(n,r)=\frac{1}{r!}\begin{vmatrix}\Sigma_1 & -1 & 0 & 0 & \cdots & 0\\ \Sigma_2 & \Sigma_1 & -2 & 0 & \cdots & 0\\ \Sigma_3 & \Sigma_2 & \Sigma_1 & -3 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \Sigma_r & \Sigma_{r-1} & \Sigma_{r-2} & \Sigma_{r-3} & \cdots & \Sigma_1\end{vmatrix} \qquad \ldots (6)$$

(*b*) *Expression for $S(n, r)$.*

Consider now the case where repetition of the same letter is not allowed. The sum of homogeneous products of r dimensions is simply the sum of the products of the n quantities taken r at a time

We take instead of (1) the expression

$$f(x)=(1-\alpha x)(1-\beta x)\cdots(1-\mu x)$$
$$=1-S_1x+S_2x^2-\cdots+(-1)^rS_rx^r+\cdot+(-1)^n S_nx^n \qquad \ldots (7)$$

and treat it exactly as before, we obtain

$$S(n,r)=\Delta(n,r)\div r! \qquad \ldots (8)$$

We may note here for future reference that if we start with the equations (2) and eliminate the quantities $\Sigma_1, \Sigma_2, \cdots \Sigma_{r-1}$ we obtain instead of (6) the result

$$\Sigma_r=(-1)^r\begin{vmatrix}H_1 & 1 & 0 & 0 & \cdots & 0\\ 2H_2 & H_1 & 1 & 0 & \cdots & 0\\ 3H_3 & H_2 & H_1 & 1 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot\\ rH_r & H_{r-1} & H_{r-2} & H_{r-3} & \cdots & H_2\end{vmatrix} \qquad \ldots (6')$$

Similarly with (8) we get the result

$$\sum_r = (-1)^r \begin{vmatrix} -S_1 & 1 & 0 & 0 & \cdots & 0 \\ 2S_2 & -S_1 & 1 & 0 & \cdots & 0 \\ -3S_3 & S_2 & -S_1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ (-1)^r r S_r & (-1)^{r-1} S_{r-1} & \cdot & \cdot & \cdots & -S_1 \end{vmatrix} \quad \ldots \quad (8')$$

3. Symmetric Functions of roots of an equation.

(*a*) Identities involving the co-efficients and sums of powers of roots.

Let us assume that $\alpha, \beta, \gamma, \ldots \mu$ considered above are the roots of the equation

$$\phi(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0. \quad \ldots \quad (9)$$

so that $\phi(x) = (x-\alpha)(x-\beta) \cdots\cdots (x-\mu)$

and $$x^n \phi\left(\frac{1}{x}\right) = (1-\alpha x)(1-\beta x) \cdots\cdots (1-\mu x). \quad \ldots \quad (10)$$

Also by (9),

$$x^n \phi\left(\frac{1}{x}\right) = 1 + p_1 x + p_2 x^2 + \cdots + p_n x^n. \quad \ldots \quad (11)$$

Now (11) is the same as (7) above and hence comparison between (11) and (7′) gives $p_r = (-1)^r\, S(n, r)$ so that from (8) we get

$$p_r = (-1)^r\, \Delta(n, r) \div r! \quad \ldots \quad (12)$$

giving the coefficients of the equation in terms of sums of powers of roots.

Also conversely as in (8′),

$$\sum_r = (-1)^r \begin{vmatrix} p_1 & 1 & 0 & 0 & \cdots & 0 \\ 2p_2 & p_1 & 1 & 0 & \cdots & 0 \\ 3p_3 & p_2 & p_1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ rp_r & p_{r-1} & p_{r-2} & p_{r-3} & \cdots & p_1 \end{vmatrix}$$

4. (*a*) **Two Properties of $\Delta(n, r)$ and a particular case.**

It is obvious from definition that

$$S(n, n+j) \equiv 0 \qquad (j = 1, 2, 3, \ldots .)$$

$$\dot{S}(n, n) = \alpha\, \beta\, \gamma \ldots \mu.$$

Hence it follows from (8) that

(i) $\Delta(n, n+j) \equiv 0 \quad (j=1, 2, 3, \ldots)$ ... (13)

(ii) $\dfrac{\Delta(n, n)}{\alpha\beta\gamma \cdots \mu}$ is always an integer, being in fact equal to $n!$ ... (14)

The case where $\alpha=1, \beta=2, \gamma=3, \ldots \mu=n$ is of special interest.

(i) Since $S(n, r)$ is always integral it follows from (8) that $\Delta(n, r)$ is a multiple of $r!$ $(r<n)$. ... (15)

(ii) Again $\Delta(n, n)$ is a perfect square, and is equal to $(n!)^2$ ... (14a)

(iii) Further $S(n, n+j)$ is zero for $j=1, 2, 3, \ldots$
Hence $\Delta(n, n+j) \equiv 0 \quad (j=1, 2, 3, \ldots)$ ... (13a)

(*b*) **A determinant involving the factorial function $n^{(r)}$.**

The well-known method for the summation of the series $\sum_1^m n^p$ by means of factorial functions gives rise to somewhat similar determinantal identities involving factorials.

If $n^{(r)} \equiv n(n+1)(n+2) \cdots (n+r-1)$, we set

$$n^p = A_1 n^{(1)} + A_2 n^{(2)} + \cdots + A_r n^{(r)} + \cdots + A_p n^{(p)} \cdots \qquad \ldots (16)$$

and perform the summation term by term using the result

$$\sum_1^m n^{(r)} = \left[\frac{n^{(r+1)}}{r+1}\right]_0^m = \frac{m^{(r+1)}}{r+1}.$$

To evaluate the constants $A_1, A_2, \ldots A_p$ we substitute in (16) $n=-1, -2, -3, \ldots -p$ in succession and noting

$$(-n)^{(r)} = (-1)^r n^{(r)}, \text{ if } r \leqslant n$$
$$= 0 \qquad , \text{ if } r > n$$

we get a set of linear equations the solution of which gives

$$(-1)^{r-1}(1!\,2!\,3!\ldots r!)A_r = \begin{vmatrix} 1^p & 1^{(1)} & 0 & 0 & \ldots & 0 \\ 2^p & 2^{(1)} & 2^{(2)} & 0 & \ldots & 0 \\ 3^p & 3^{(1)} & 3^{(2)} & 3^{(3)} & \ldots & 0 \\ r^p & r^{(1)} & r^{(2)} & r^{(3)} & \ldots & r^{(r-1)} \end{vmatrix} = D(n, r), \text{ say}, \qquad \ldots (17)$$

In particular it is obvious that $A_p = 1$ and $A_{p+j} \equiv 0$ ($j = 1, 2, 3, \ldots$) Hence we have, if p is any integer,

$$\left.\begin{array}{l} |D(p,p)| = 1!\,2!\,3!\ldots p! \\ D(p, p+j) \equiv 0 \quad (j = 1, 2, 3, \ldots) \end{array}\right\} \tag{18}$$

Further it is obvious that the results (18) are true even if we replace the first column in (17) by $\phi_p(1), \phi_p(2), \ldots \phi_p(r)$ where $\phi_p(n)$ is any polynomial of degree p in n, such as

$$\phi_p(n) = n^p + a_1 n^{p-1} + \cdots + a_p.$$

5. Reversible Determinants.

The equations (13) and (13′) suggest the possibility of what may be called Reversion of Determinants. It can be easily shown that if two sequences ($a_1, a_2, a_3, \ldots$) and ($b_1, b_2, b_3, \ldots$) are such that

$$k_{r,0}\, a_r + k_{r-1,1}\, a_{r-1} b_1 + k_{r-2,2}\, a_{r-2} b_2 + \cdots + k_{0,r}\, b_r = 0$$

for $r = 1, 2, 3, \ldots$ then a_r and b_r can be expressed as determinants that are mutually reversible. In other words, any element a_r of the set $[a]$ can be expressed as a determinant of the rth order in $b_1, b_2, \ldots b_r$ and, conversely. The results of (13) and (13′) show that the sets formed by the sum of the rth powers of the roots and the coefficients of a polynomial equation are mutually reversible.

An illustration of reversibility which is also symmetric is given below :

If
$$a_r = \begin{vmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 \\ b_2 & b_1 & 1 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 \\ b_r & b_{r-1} & b_{r-2} & b_{r-3} & \cdots & b_1 \end{vmatrix},$$

then
$$b_r = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_r & a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_1 \end{vmatrix}$$

for $r = 1, 2, 3, 4, 5, \ldots$ The proof follows at once on expansion of either of the two determinants.

The case where a_r and b_r are equal is of special interest. The determinantal expressions for a_r and b_r become identical and either of them may be said to be Self-reversible. An example of a self-reversible determinant is

$$\frac{1}{r!} = \begin{vmatrix} \frac{1}{1!} & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \frac{1}{2!} & \frac{1}{1!} & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \frac{1}{(r-3)!} & \cdot & \cdot & \cdot & \cdot & \frac{1}{1!} \end{vmatrix} \dagger$$

† See Qn. 1397, J. I. M. S., Vol. XVII, 1928, p. 119.

GLEANINGS

The *Timaeus* is a fascinating anthology of the queer perversities to which this magic of symbolism could be pushed. Real earth, as opposed to the solid earth on which we build houses, is an equilateral triangle. Real water, as opposed to what is sometimes regarded as a beverage, is a right angled triangle. Real fire as opposed to fire against which you insure, is an isosceles triangle. Real air as opposed to the Air which you pump into a tyre is a scalene triangle. Lest you should find this hard to credit, read how Plato turned the geometry of the sphere into a magical explanation of man's origin. God, he tells us, "imitating the spherical shape of the universe enclosed the two divine courses in a spherical body, that, namely, which we now term the head." In order that the head "might not tumble about among the deep and high places of earth, but might be able to get out of the one and over the other," it was provided with "the body to be a vehicle and means of locomotion, which consequently had length and was furnished with four limbs extended and jointed......" This supremacy of the head is very flattering to intellectuals who have no practical problems to occupy them............ An educational system which was based on Plato's teaching is apt to entrust the teaching of mathematics to people who put the head before the stomach, and who would tumble about the deep and high places of the Earth if they had to teach another subject.

From HOGBEN; *Mathematics for the Million.*

THE LOGIC OF INTUITIONISTIC MATHEMATICS

By

K. Chandrasekharan, *University of Madras.*

§1. Intuitionistic Mathematics, which was for the first time advanced and systematized by Brouwer,* is based on the rejection of the unlimited validity of the principle of the excluded middle, and the consequent emphasis laid on the constructive character of mathematical concepts. This leads to revolutionary changes in every branch of Mathematics in general, and set-theory in particular. Intuitionists, for instance, characterize the classical definition of set as vague, and hence inadmissible; they do not permit the statement that every set is finite or infinite. The new concept of set which they propound is based, as they would say, on time-consciousness (zeit-intuition) itself, which they consider primordial. The resulting set-theory, though more complicated than the classical one, can be said to be more complete in as much as it takes cognisance of possibilities to which the classical theory gave little thought.

The object of this paper is to give an account of the basic concepts of intuitionistic set-theory, and study them in the light of the peculiar features of intuitionistic logic.

§2. *The Propositional Calculus.*

The propositional calculus, viewed from the logical stand-point, is the study of the theory of relations between elementary propositions, expressed by the words 'or', 'and', 'not', 'if' and denoted respectively by $+$, $\times$, $\neg$, $\rightarrow$. These four relations, however, are not independent of one another. We can take, for instance, $\neg$, $+$ as basic, and define the other two thus:

$$p \times q \equiv \neg(\neg p + \neg q)$$
$$p \rightarrow q \equiv \neg p + q$$

The meaning of 'not' is fixed by the following laws: †

L_1. Law of Contradiction
$\neg a$ is logically inconsistent with a.

L_2. Any proposition contradicting a implies $\neg a$.

L_3. Law of the Excluded Middle.
Either a or $\neg a$ is true.

L_4. Law of double-negation.
$\neg\neg a$ is the same as a.

* Brouwer (2)

† Pankajam (11)

If, on the other hand, we take the relations $+$, $\times$, $\neg$, $\rightarrow$ as fundamental and not definable in terms of one another, and fix the meaning of negation so as to satisfy only L_1 and L_2, the resulting calculus of propositions is termed Brouwerian or Intuitionistic; to render this calculus consistent, the relations of $+$, $\rightarrow$ also undergo a corresponding change in meaning. Thus, when we say $a+b$ is provable, we mean a is provable, or b is provable; and we have to *specify* which of the two is provable. In this sense, $a+b$ is not equivalent to $\neg a \rightarrow b$. The specifiable nature associated with the relation of 'or' is illustrated by the fact, that in the Intuitionistic Mathematics we cannot prove the theorem: †

If a, b, are real numbers, and $ab=0$, then either $a=0$ or $b=0$. We can only say, instead,

If $ab=0$, then $a \neq 0$, $b \neq 0$, cannot hold simultaneously, from which we are not allowed to infer that either $a=0$, or $b=0$, since we cannot always assert the truth of a *specifiable* one of the two alternatives. It is important to observe that the demand on the constructive character of mathematics is an extension of this meaning of 'or'. The meaning of '$\rightarrow$' also undergoes a change, in as much as only the following proposition holds, and not the converse:

$$(a \rightarrow b) \rightarrow \neg (a \times \neg b).$$

But, if b is 'normal' *i.e.*, $\neg \neg b = b$, then,

$$(a \rightarrow b) \equiv \neg (a \times \neg b),$$

which we shall shew presently.

To distinguish the new relations from the old, let us denote them by $\vee$, $\wedge$, $\supset$, $\urcorner$, which stand respectively for 'or,' 'and,' 'if,' 'not.'

The relationship between the classical and Brouwerian logics has been established by Glivenko ‡ and Gödel in the following theorems.

I. If the formula a is provable in the propositional calculus of classical logic, then $\urcorner\urcorner a$ is provable in intutionistic logic. [Glivenko].

$$\text{Ex.} \quad \vdash : \urcorner\urcorner (\urcorner\urcorner a \supset \cdot a)$$
$$\vdash : \urcorner\urcorner (a \cdot \vee \cdot \urcorner a)$$

i.e. Every true proposition of classical mathematics is 'half-true' in intuitionistic mathematics. A half-true proposition is one whose double negation is true.

† Heying (5)

‡ Glivenko (8).

Ex.: If a, b, are real numbers, and $ab=0$, then $a=0$ or $b=0$ is 'half-true' in intuitionistic mathematics.

II. If we interpret the classical relations

$$\neg p, \quad p \rightarrow q, \quad p+q, \quad p \times q$$

to mean $\urcorner p, \ \urcorner(p \cdot \wedge \cdot \urcorner q), \ \urcorner(\urcorner p \cdot \wedge \cdot \urcorner q), \ p \cdot \wedge \cdot q,$

then every formula valid in the classical logic is also valid in intuitionistic logic. [Gödel].

Ex. $a + \neg a$, which is provable in classical mathematics, becomes $\vdash : \urcorner(\urcorner a \cdot \wedge \cdot \urcorner\urcorner a)$.

It should be noted, however, that though 'or' has a diffrren. shade of meaning in intuitionistic logic, this does not affect its formal properties such as $a \supset \cdot a+b$, $a+b=b+a$, $a(b+c)=ab+ac$ etc.

The application of lattice-theory to the study of the propositional calculus * lends a clearer view of the structure of the propositional calculus as affected by the different meanings that we attribute to negation in the classical and the intuitionistic logics. It is known that the propositions from a distributive lattice under the lattice-operations $+$, $\times$, standing for 'or' and 'and,' and that this lattice contains units 0 and 1, which are identifiable respectively with those propositions which are 'absurd' and those which are 'logically necessary.' The implication-relation can be identified with the ordering-relation in the lattice. If, in addition, each element admits a complement, which can be identified with the logical relation of negation, the set of propositions forms a Boolean Algebra. The study of this Boolean Algebra amounts, then, to the study of the classical proportional calculus.

If, on the other hand, we assume that each element of the distributive propositional lattice admits a product-complement, this product-complement turns out to be the same as the relation of negation in Brouwerian logic. This important result is due to Miss S. Pankajam. The properties of the product-complement are carried over into formulae in intuitionistic logic, as we shall show now.

In the theory of the product-complement we have the following:

Defn.: An element a is said to be 'Simple,' if $a+a'=1$.

Defn.: An element a is said to be 'Normal' if $a=a''$.

* S. Pankajam (II).

Also, we have,

$$(a+b)'=a'b' \tag{2·1}$$

$$\text{If } a < b, \text{ then } a' > b' \tag{2·2}$$

$$a < a''; \quad a'''=a' \tag{2·3}$$

$$(ab)'=(a'+b')'' \tag{2·4}$$

For proofs of (2·1), (2·2), (2·3) see S. Pankajam [11]. (2·4) can be proved as follows:

$ab < a$, $ab < b$. and so, $(ab)' > (a'+b')$ or, $(ab)' > (a'+b')''$. Next, $(ab)'(a'+b')'=(ab+a'+b')'=[(a+a'+b')(b+b'+a')]'$, which is $< [(a+a')(b+b')]'$. Now, $(a+a')'=0$, and $(b+b')'=0$. If $x'=0$, $y'=0$, then we can shew $(xy)'=0$. For, if $xz=0 \supset. z=0$ and $yz=0 \supset. z=0$, then $x(yz)=0 \supset. yz=0 \supset. z=0$, so that $(xy)'=0$. Applying this result we get $[(a+a')(b+b')]'=0$.

Hence $(ab)' < (a'+b')''$, which proves the result.

Correspondingly in logic we have,

$$\urcorner(a.\vee.b):\supset\subset:\urcorner a.\wedge.\urcorner b \tag{2·5}$$

$$a\supset b:\supset:\urcorner b\supset.\urcorner a \tag{2·6}$$

$$a\supset\urcorner\urcorner a;\ \urcorner\urcorner\urcorner a:\supset\subset:\urcorner a \tag{2·7}$$

$$\urcorner(a.\wedge.b):\supset\subset:\urcorner\urcorner(\urcorner a.\vee.\urcorner b) \tag{2·8}$$

For proofs of (2·5) to (2·7) see Heyting [5]·(2·8) can be proved as follows:

Firstly, $\urcorner(\urcorner a.\vee.\urcorner b)\equiv\urcorner\urcorner a.\wedge.\urcorner\urcorner b$ by (2·5)
And, $a.\wedge.b\supset.\urcorner\urcorner a.\wedge.\urcorner\urcorner b$
$\therefore$ $a.\wedge.b\supset.\urcorner(\urcorner a.\vee.\urcorner b)$
i.e. $\urcorner\urcorner(\urcorner a.\vee.\urcorner b)\supset.\urcorner(a.\wedge.b)$

Secondly,

$$\urcorner(a.\wedge.b):\supset:a\supset\urcorner b.\wedge.b\supset.\urcorner a$$
$$a\supset\urcorner b.\wedge.b\supset.\urcorner a:\supset:\urcorner\urcorner b\supset.\urcorner a.\wedge.\urcorner\urcorner a\supset\urcorner b.$$
$$\urcorner\urcorner b\supset.\urcorner a.\wedge.\urcorner\urcorner a\supset.\urcorner b:\supset:\urcorner\urcorner b.\wedge.\urcorner\urcorner a\supset.\urcorner a.\wedge.\urcorner b.$$
$$\urcorner a.\wedge.\urcorner b:\supset:\urcorner a.\vee.\urcorner b.$$

Hence, $\urcorner\urcorner b.\wedge.\urcorner\urcorner a:\supset:\urcorner a.\vee.\urcorner b.$
i.e. $\urcorner(\urcorner a.\vee.\urcorner b):\supset:\urcorner a.\vee.\urcorner b.$
i.e. $\vdash:\urcorner\urcorner(\urcorner a.\vee.\urcorner b).$
Hence, $\urcorner(a.\wedge.b):\supset:\urcorner\urcorner(\urcorner a.\vee.\urcorner b).$

We can now easily prove that,

$$a\supset b:\supset\subset:\urcorner(a.\wedge.\urcorner b) \text{ if } b \text{ is normal.} \tag{2·9}$$

Proof:—For this we shew that whether b is normal or not, the right side $\equiv a \supset \neg\neg b$.

If $a \supset b$, then $(a. \wedge. \neg b) : \supset : (b. \wedge. \neg b)$ which is absurd.

So, $a \supset b : \supset : \neg (a. \wedge. \neg b)$, and,

$a \supset b. \supset. \neg (a. \wedge. \neg b) : \supset : a \supset \neg\neg b. \supset. \neg (a. \wedge. \neg b)$.

So, $a \supset \neg\neg b : \supset : \neg (a. \wedge. \neg b)$.

Conversely, $\neg (a. \wedge. \neg b) = \neg\neg (\neg a. \vee. \neg\neg b) = \neg (\neg\neg a. \wedge. \neg b)$.

$\neg (\neg\neg a. \wedge. \neg b) : \supset : \neg\neg a \supset. \neg\neg b$, and

$\neg\neg a \supset. \neg\neg b : \supset : a \supset \neg\neg b$.

Hence, $a \supset \neg\neg b \equiv \neg (a. \wedge. \neg b)$.

§3. *The Function Calculus.*

The classical function-calculus deals with analysed propositions, and can be developed as an axiomatic system in the same way as the calculus of elementary propositions. The two basic types of propositional functions are the 'universal' and the 'particular' together with their negations. A universal proposition is of the form, 'for all x, F (x)' denoted by (x) F (x), while a 'particular' proposition is of the form 'exists an x such that F (x)' denoted by (Ex) F (x). The fundamental propositions in the classical function-calculus are

$$(Ex)\ F(x) \rightleftarrows \overline{(x)}\ \overline{F(x)} \qquad (3\cdot1)$$

$$(Ex)\ \overline{F(x)} \rightleftarrows \overline{(x)}\ F(x). \qquad (3\cdot2)$$

The intuitionistic function-calculus is similarly developed from the intuitionistic propositional calculus, which, in its turn, differs from the classical one in its rejection of the principle of the excluded middle. It is natural to expect, therefore, that both the above formulae will not hold in the new calculus. The amended forms which are valid will be established presently. Gödel has proved the following theorem:*

III. If the operations of the classical calculus of propositions are translated in the manner stated in Th. II, and if $(x)\ a(x)$ is translated as $(x)\ a(x)$, and (Ex) $a(x)$ as $\neg(x)\ \neg a(x)$, then every classical formula of the function-calculus goes over into a valid formula in intuitionistic function-calculus.

Ex: (3.1) becomes $\vdash : \neg [\neg (x) \neg F (x). \wedge. \neg\neg (x) \neg F (x)]$ which can be seen directly from the Law of contradiction.

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* Heyting (5)

We can also seek to apply the theory of the distributive lattice so as to establish some of the formulae in function-calculus, if we remember that $(x)\ F(x)$ stands for the logical product of a series of propositions, and $(Ex)\ F(x)$ for their logical sum. In the case of this continued sum which involves endless disjunction, a definite one of the many possible alternatives must be specifiable, and this specification of the meaning of 'or' satisfies the demand on the constructive character of Mathematics which the intuitionists emphasize. The product and sum respectively exist, since the quantised propositions are significant. We can observe the correspondence clearly by establishing the following formulae from purely lattice-considerations involving product-complements of unrestricted (existing) sums and products. Their logical proofs are to be found in Heyting's paper.*

$$\vdash : (x) \urcorner F(x) : \supset\subset : \urcorner (Ex)\ F(x) \qquad (3\cdot3)$$

$$\vdash : (x)\ F(x) : \supset : \urcorner (Ex) \urcorner F(x) \qquad (3\cdot4)$$

$$\vdash : (Ex) \urcorner F(x) : \supset : \urcorner (x)\ F(x) \qquad (3\cdot5)$$

$$\vdash : (Ex)\ F(x) : \supset : \urcorner (x) \urcorner F(x) \qquad (3\cdot6)$$

$$\vdash : (Ex) \urcorner\urcorner F(x) : \supset : \urcorner\urcorner (Ex)\ F(x) \qquad (3\cdot7)$$

$$\vdash : \urcorner\urcorner (x)\ F(x) : \supset : (x) \urcorner\urcorner F(x). \qquad (3\cdot8)$$

To prove these in the lattice, we make use of the following result of Macneille,† viz.,

If $\sum a_i$ exists, and each a_i admits a product-complement then πa_i exists, and $(\sum a_i)' = \pi a'_i$ (3·9)

This result is an extension of (2·5) to the case of unrestricted sums, and is the same as (3·3).

Again, we have, $\pi p_i < \pi p''_i = (\sum p'_i)'$, (3·10)
which is the same as (3·4).

$(\sum p'_i) < (\sum p'_i)'' = (\pi p''_i)' < (\pi p_i)'$ (3·11)
which proves (3·5).

$(\sum p_i) < (\sum p_i)'' = (\pi p'_i)'$ which proves (3·6). (3·12)

$\sum p''_i < (\sum p''_i)'' = (\pi p'_i)'$, (3·13)
which proves (3·7)

$(\pi p_i)'' < (\sum p'_i)' = \pi p''_i$, which proves (3·8) ... ... (3·14)

* Heyting (6)
† Macneille (10)

The reverse implications of (3.7) and (3.8) do not hold, and so in a general distributive lattice we cannot assert that

$(\pi a_i)' = (\sum a_i')''$, where the product and the sum are unrestricted. However;

If every a_i is normal, then $(\pi a_i)' = (\sum a_i')''$. (3.15)

For, $(\pi a_i)' = (\pi a_i'')' = (\sum a'_i)''$

From (3.4) and (3.5) we observe that $(x)F(x)$ and $(Ex)\urcorner F(x)$ are two types of propositions each of which implies the negation of the other.

We know in general that, if $ab=0$, then $a''\ b''=0$, and $(a'+b')''=1$.

Hence, we have,

$$\vdash : \urcorner[(x)F(x) . \wedge . (Ex)\urcorner F(x)]$$
$$\vdash : \urcorner\urcorner[\urcorner(x)F(x) . \vee . \urcorner(Ex)\urcorner F(x)] \qquad (3.16)$$

§4. *Fundamental Concepts of Intuitionistic Set-Theory.**

(i) **Sets:** A *Set* is a law, such that, when again and again an arbitrary number is chosen, each of these choices either generates a definite number-sequence, or leads to a blocking of the process together with a nullification of its results, and such that, for $n > 1$, after an unended and unblocked sequence of $n-1$ choices, at least one number can be given, which, if chosen as the nth number, does not lead to a blocking of the process.

The number-sequence resulting from this *choice-sequence* (Wahlfolge) is called an *element of the set.*

If ζ is the sequence of positive integers, and corresponding to any number in ζ, a number k_n is defined in such a way that if we choose for the nth choice a number in ζ greater than k_n the process is blocked, then the set is called *finitary.*

Two set-elements are said to be *equal* or *identical* if we are certain that, for each n, the nth choice in both the elements generates the same number-sequence.

Two sets are said to be *equal* or *identical* if to each element of one, an equal element of the other can be given.

A set M is said to be a *subset* of the set N, if to every element of M, there exists an equal element of N.

Sets and set-elements are called *mathematical entities.*

* Brouwer (4) and (3).

(ii) **Species**: A *species of the first order* is a property which only a mathematical entity can possess; the entity will then be called *element of the species of first order.*

Set is a special case of species of the first order. Two *species* are said to be *equal* or *identical* if to any element of one, an equal element of the other can be given.

By a *species of the second order* is meant a property which can be possessed only by a mathematical entity or a species of the first order.

Species of higher orders can be similarly defined. The species M is called a *subspecies* of N if to each element of M, there exists an equal element of N. If in addition, an element of N can be given which is not equal to any element of M, then M is called a *proper subspecies* of N.

Two set-elements are said to be *different* if the impossibility of their identity is established.

Two *species* are *different* if the impossibility of their identity is established.

A species is said to be *discrete,* of which any two elements are either identical or different.

A species M 'projects out of' N in Black's phrase,* (herausragt) if N has an element which is different from every element of M.

Two species are said to be *congruent* if neither of them can project out of the other.

If M is a subspecies of N which is congruent to N, then M is *half-identical* with N.

The species of elements which belong either to the species M or to the species N is called the *union* of M and N, denoted by M + N.

The species of elements which belong to both the species M and N, is called the *intersecton* of M and N, denoted by M.N.

Two species M and N are *disjoint,* if they are different, and if it is impossible for a common element to exist.

If M′ and M″ are disjoint subspecies of N, and M′ + M″ is half-identical with N, then N is said to be *composed of M′ and M″*, and M′, M″ are *complementary species* of one another in N.

* Black (I)

If M′, M″ are disjoint subspecies of N, and M′+M″ is identical with N, then N is said to be separated into M′ and M″, and M′, M″ are called *conjugate separable species* of N, while M′, M″ are called *separable subspecies* of N.

If we can establish a one-to-one correspondence between two sets M and N i.e. a law by which to every element of M there corresponds an element of N (equal elements of M corresponding to equal elements of N) and every element of M corresponds to some element of N, then we say that M is *equipollent* to N (gleichmächtig), or M has the same power as N, or M ⁓ N.

A set is *finite* if it is equipollent to an initial segment of ζ. It is said to be *infinite* if it contains a subset equipollent to ζ. It is *reducibly infinite* if the concerned subset is separable.

(iii) **Examples:**

(*a*) Let M be the set of all infinite binary decimals in which we know how many 0's occurs before the first 1, or there is no 1 at all; and N the set of all infinite binary decimals in which we know how many 1's occur before the first 0 or there is no 0 at all. Then M and N are *congruent*. For, neither of the two sets can contain an element different from every element of the other. It is to be noted that we cannot assert that M is a subset of N or N is a subset of M. For, if we take an element of M which begins with 1, we cannot produce an equal element of N, or, in other words, the set-law of N does not enable us to assert that it is an element of N. Though we cannot thus say that either of them is a subset of the other, it is absurd to say that either of them contains an element which 'differs' from every element of the other.

(*b*) Let M be the same as above, and N the set of all infinite binary decimals. Then M is *half-identical* with N. For, from (*a*), we observe they are congruent, and since every element of M is an infinite binary decimal, and N contains all the infinite binary decimals, M is a subset of N.

(*c*) Let N be the set of all natural numbers, M′ those which, as exponents, make Fermat's equation $x^n + y^n = z^n$ soluble in positive integers x, y, z, and M″ those which, as exponents, makes the equation impossible to solve. Then N is composed of the *complementary species* M′ and M″.

It is clear that M′ and M″ are disjoint subsets of N, so that M′+M″ is a subset of N. Since M′+M″ is a subset of N, it is

absurd to say that $M'+M''$ contains an element different from every element of N. Again, we cannot assert that N is a subset of $M'+M''$; for, given any integer, we cannot verify in a finite number of steps whether or not it makes Fermat's equation soluble. Nevertheless it is absurd to say that N contains an element which is 'different' from every element of $M'+M''$. For, given an integer, it is absurd to say that it is neither in M' nor M''. This is an instance of the surviving form of the principle of the excluded middle, namely, $\vdash : \urcorner(\urcorner a \cdot \wedge \cdot \urcorner\urcorner a)$.

(*d*) Let N be the set of all natural numbers, M' those which have at most five digits, and M'' those which have more than five digits. Then N is *separated* into M' and M''.

For, M' and M'' are disjoint subsets of N, and therefore $M'+M''$ is a subset of N; next, N is also a subset of $M'+M''$, for, given any integer in N, we can certainly produce an equal integer in $M'+M''$: or, in other words, specify to which of the M', M'' it belongs.

§5. *Some Remarks*:

From the examples cited above, it must be clear that we cannot always assert that a set is either finite or infinite. For instance, the set of integers which, as exponents, make Fermat's equation soluble cannot be asserted to be either finite or infinite.

Let $M < N$, denote that M is a *subset* of N.

Let $M * \not< N$, denote that M is *positively not a subset* of N i.e., N 'projects out' of M. Then, $M < N$ is a proposition of the form $(x)\, a_x$, and $M * \not< N$ is a proposition of the form $(Ex)\urcorner a_x$. Then, from (3.4), (3.5) and (3.15) we observe that

(i) 'Subset' and 'positively not a subset' are two relations which contradict each other, and therefore their double negations also are mutually contradictory.

(ii) 'M is a subset of N, or M is positively not a subset of N' is a 'half-true' proposition i.e. a proposition whose double negation is true.

If M and N are identical, then $M < N$, $N < M$, so that we have $\urcorner(M * \not< N) . \wedge . \urcorner(N * \not< M)$ which means that

(iii) identity always implies congruence.

The relation that 'm is an element of M' is neither 'simple' nor 'normal'. That it is not simple can be easily seen from the example

that given an integer we cannot always say that it makes Fermat's equation *either* soluble *or* not (in the intuitionist meaning of 'or'). That it is not normal can be deduced from the fact that the assertion of existence is not, in general, identical with its double negation, i.e. we cannot maintain that $(\mathrm{E}x)a_x = \urcorner\urcorner(\mathrm{E}x)a_x$.

Similarly, the relations, 'subset' and 'project out' are neither simple nor normal.

'Congruence' is a normal relation, since it is the product of two negations. We cannot assert, in general, that it is simple.

The existence of normal elements which are not simple in the lattice is testified by the existence of normal relations in intuitionistic mathematics which are not simple. An easy example can be given from the real-number theory.*

A real number is defined by a λ-sequence of intervals, each of which is contained completely in the preceding, given by

$$\left[\frac{a}{2^{n+1}}, \frac{(a+2)}{2^{n+1}}\right],$$

where a is an integer.

If in every interval of the real number A, there is an interval of the real number B, and vice-versa, then the two numbers are said to be equal, $A = B$.

Two numbers A, B, are said to be *positively different* if an interval of A and an interval of B can be given, which are disjoint. Then $A{*}\neq{*}B$,

Two numbers are said to be *different* if their equality leads to a contradiction. $A \neq B$.

It has been shown by Heyting that $\urcorner(A{*}\neq{*}B) \equiv (A = B)$. Therefore, $A = B$ is a normal relation, for $\urcorner\urcorner(A = B) \equiv \urcorner\urcorner\urcorner(A{*}\neq{*}B)$

$$\equiv \urcorner(A{*}\neq{*}B) \equiv (A = B)$$

But $A = B$ is not a 'simple' relation, in general, as can be seen from the following example:

Consider the decimal expansion for π, and let k be the rank of the digit at which the sequence 0123456789 begins. Define a sequence (a_n) thus: $a_n = 2^{-n}$ if for the first n digits there is no such k; if there is such a k, and it is odd, then also, $a_n = 2^{-n}$; if there is such a k, and it is even, then $a_n = 2^{-k}$. Let the limit of this

* Heyting (5).

sequence (a_n) be denoted by A. Then, we cannot say that a *specifiable* one of the two relations $A=0$, or $A\neq 0$ must necessarily hold.

I cannot conclude without saying how thankful I am to Dr. R. Vaidyanathaswamy, not only for his departmental lectures on Intuitionism, but for his kind encouragement and valuable criticism.

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NOTE ON THE LIMIT OF CORRELATION AND REGRESSION COEFFICIENTS IN MINGLED RECORDS

BY

D. P. BANERJEE, D.Sc.

In the present note I shall find the limits of regression coefficients when a number of statistical records are mingled together. I shall also find the limit and behaviour of inter class correlation coefficient.

Let there be k records and x and y be two variables. In the sth record let N_s, M_{sx}, M_{sy}, σ_{sx}, σ_{sy}, r_s, $b_s \left(= r_s \frac{\sigma_{sy}}{\sigma_{x}}\right)$ be the number of observations, means, the standard deviation, the correlation coefficient and regression coefficient respectively. Let the corresponding constants when the records are mingled be N, M_x, M_y, σ_x, σ_y, r, $b \left(= r \frac{\sigma_y}{\sigma_x}\right)$ respectively.

$$\text{Then } N = \sum_{s=1}^{k} N_s$$

$$N M_x = \sum_{s=1}^{k} N_s M_{sx}, \quad N M_y = \sum_{s=1}^{k} N_s M_{sy}$$

$$N \sigma_x^2 = \sum_{s=1}^{k} N_s \sigma_{sx}^2 + \sum_{s=1}^{k} N_s (M_{sx} - M_x)^2$$

$$N \sigma_y^2 = \sum_{s=1}^{k} N_s \sigma_{sy}^2 + \sum_{s=1}^{k} N_s (M_{sy} - M_y)^2$$

$$r = \frac{\sum_{s=1}^{k} N_s r_s \sigma_{sx} \sigma_{sy} + \sum_{s=1}^{k} N_s (M_{sx} - M_x)(M_{sy} - M_y)}{N \sigma_x \sigma_y}$$

$$b = r \frac{\sigma_y}{\sigma_x} = \frac{\sum_{s=1}^{k} N_s b_s \sigma_{sx}^2 + \sum_{s=1}^{k} N_s (M_{sx} - M_x)(M_{sy} - M_y)}{\sum_{s=1}^{k} N_s \sigma_{sx}^2 + \sum_{s=1}^{k} N_s (M_{sx} - M_x)^2}$$

Let R, B and S_x *be the corresponding statistical constants corresponding to the* sample formed by the pairs of Mean $(M_{1x}, M_{1y}) \ldots (M_{kx}, M_{ky})$ with frequencies $N_1, N_2 \ldots, N_k$ respectively.

Then $b = \dfrac{\sum_{s=1}^{k} N_s\, b_s\, \sigma_{sx}^2 + NB\, S_x^2}{\sum_{s=1}^{k} N_s\, \sigma_{sx}^2 + NS_x^2}$

Let b' be the superior limit of $b_1, b_2, \ldots b_k$, B then $b' - b$

$$= \frac{\sum_{s=1}^{k} N_s\, \sigma_{sx}^2\, (b' - b_s) + NS_x^2\, (b' - B)}{\sum_{s=1}^{k} N_s\, \sigma_{sx}^2 + NS_x^2}$$

= positive Hence $b' > b$.

Again $b - \dfrac{\sum_{s=1}^{k} N_s\, b_s\, \sigma_{sx}^2 - b'\, NS_x^2}{\sum_{s=1}^{k} N_s\, \sigma_{sx}^2 + NS_x^2} = \dfrac{N\, (b' + B)\, S_x^2}{\sum_{s=1}^{k} N_s\, \sigma_{sx}^2 + NS_x^2}$

which is positive. Hence $b > \dfrac{\sum_{s=1}^{k} N_s\, b_s\, \sigma_{sx}^2 - b'\, NS_x^2}{\sum_{s=1}^{k} N_s\, \sigma_{sx}^2 + NS_x^2}$

Therefore $\dfrac{\sum_{s=1}^{k} N_s\, b_s\, \sigma_{sx}^2 - b'\, NS_x^2}{\sum_{s=1}^{k} N_s\, \sigma_{sx}^2 + NS_x^2} \leqslant b \leqslant b'$

Let us now consider the intraclass correlation i.e. correlation between variables of identical character Intraclass correlation coefficient of the sth record

$$r_s = \frac{\sum_{s=1}^{Ns} (X_{st} - M_s)(Y_{st} - M_s)}{N_s\, \sigma_s^2}$$

the correlation coefficient after mingling is

$$r = \frac{\sum_{s=1}^{k} N_s\, r_s\, \sigma_{sx}\, \sigma_{sy} + \sum_{s=1}^{k} N_s\, (M_{sx} - M_x)(M_{sy} - M_y)}{N\, \sigma_x\, \sigma_y}$$

$= \dfrac{\sum_{s=1}^{k} N_s\, r_s\, \sigma_s^2 + N_s^2}{\sum_{s=1}^{k} N_s\, \sigma_s^2 + N_s^2}$ Since $M_{sx} = M_{sy} = M$.

$\sigma_{sx} = \sigma_{sy} = \sigma^s$,

Let r' be the superior limit of $r_1, r_2 \cdots r_k$

$$\text{Then } |r| \leqslant \left| \frac{\sum_{s=1}^{k} N_s r_s \sigma_s^2 + N_s^2}{\sum_{s=1}^{k} N_s \sigma_s^2 + N_s^2} \right|$$

$$\leqslant \frac{\left| \sum_{s=1}^{k} N_s \sigma_s^2 r_s \right| + N_s^2}{\sum_{s=1}^{k} N_s \sigma_s^2 + N_s^2} < \frac{r' \sum_{s=1}^{k} N_s \sigma_s^2 + N_s^2}{\sum_{s=1}^{k} N_s \sigma_s^2 + N_s^2}$$

$$\leqslant r' + \frac{(1-r')\, N_s^2}{\sum_{s=1}^{k} N_s \sigma_s^2 + N_s^2}$$

This result is more accurate than that of Mr. B. Sitaraman*
Proceeding as Mr. Sitaraman

$$\left| \frac{\sum_{s=1}^{k} N_s r_s \sigma_s^2}{\sum_{s=1}^{k} N_s \sigma_s^2} \right| \leqslant r$$

$$\text{Hence} \quad \left| \frac{\sum_{s=1}^{k} N_s r_s \sigma_s^2}{\sum_{s=1}^{k} N_s \sigma_s^2} \right| \leqslant r \leqslant r' + \frac{(1-r')\, N_s^2}{\sum_{s=1}^{k} N_s \sigma_s^2 + N_s^2}$$

when $r_1 = r_2 = r_3 = \cdots = r_k = 0$

$$r = \frac{N_s^2}{\sum_{s=1}^{k} N_s \sigma_s^2 + N_s^2} \neq 0$$

unless $s=0$, $\sum_{s=1}^{k} N_s \sigma_s^2$ being not zero.

Hence if the intraclass correlation coefficient of each of the k records be zero, then the intraclass correlation coefficient of the mingled record will not always be equal to zero.

* B. Sitaraman: *The Mathematics Student* Vol. VIII No. 2 p. 75.

ON THE CONICOIDS OF A PENCIL TOUCHING A GIVEN PLANE

BY

V. RANGACHARIAR, *Science College, Patna.*

1. Three conicoids can, in general, be drawn through the quartic curve Γ of intersection of two given conicoids and touching a given plane π. The object of the present paper is to discuss the nature of the three conicoids. It is found that of the three conicoids, one is always real and a hyperboloid of one sheet. The nature of the other two conicoids depends on the position of the plane with reference to the quartic curve and the paraboloids of the pencil.

2. The quartic curve Γ meets the plane π (both supposed real) in four points real or imaginary. A conicoid through Γ and touching π must have the line-pairs through the four points as generators and have the vertices of the line-pairs as the points of contact with the plane. If therefore the four points are all real, so are also the line pairs and the three conicoids are all real hyperboloids of one sheet. If only two points are real, one line-pair is real giving a hyperboloid of one sheet. The other two line-pairs are imaginary with imaginary vertices and hence in this case only one conicoid of the system is real and a hyperboloid of one sheet. If the four points are all imaginary, one line-pair is real, two are imaginary with real vertices. Hence in this case one conicoid is a hyperboloid of one sheet; the other two are ellipsoids or two sheeted hyperboloids or one an ellipsoid, the other a hyperboloid of two sheets.

3. What has been said above is also true for the plane at infinity, i.e., if Γ meets the plane at infinity in four real points, the three paraboloids are all hyperbolic; if while Γ meets it in two real points, only one paraboloid is real and hyperbolic; if in four imaginary points, one paraboloid is hyperbolic and the other two elliptic. If therefore the three paraboloids are all real and hyperbolic or only one is real, Γ has an infinite branch; the two other conicoids touching π when it meets Γ in four imaginary points must be hyperboloids of two sheets. Only one case remains to be discussed when two of the paraboloids are elliptic and π does not intersect Γ. I give below an analytical demonstration of this case as I failed to obtain a proof by synthetic methods.

4. It has been shown above that it is always possible to draw hyperboloids of one sheet through Γ. Let therefore Γ be defined as the intersection of $\sum ax^2=0$ and $\sum a'x^2=0$ in quadriplanar coordinates where $a, b, a', b' > 0$ and $c, d, c', d' < 0$. The parameters of the conicoids corresponding to the plane $\sum lx=0$ are the roots of $\sum l^2/(a-\lambda a') \equiv \sum l^2/a'(\alpha-\lambda) \equiv \sum b'c'd'l^2(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda)/a'b'c'd'(\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda) \equiv F(\lambda)/a'b'c'd'(\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda)=0$. As explained above I consider the case when the roots of $F(\lambda)=0$ are all real, one determining a hyperboloid of one sheet, the other two ellipsoids or two-sheeted hyperboloids or one ellipsoid and the other a two-sheeted hyperboloid. It can be easily seen that a conicoid $\sum ax^2=0$ where $a, b, c, >0$ and $d<0$ is cut or is not cut by the plane $\sum lx=0$ in a real section according as $\sum l^2/a \lesseqgtr 0$. It is a hyperboloid of two sheets or an ellipsoid according as $\sum(1/a) \lesseqgtr 0$. Hence a value θ of λ which makes three of $a'(\alpha-\theta)$, $b'(\beta-\theta)$, $c'(\gamma-\theta)$, $d'(\delta-\theta)$ positive and the fourth negative gives an ellipsoid or a hyperboloid according as $\sum{}^1/a'(\alpha-\theta) \equiv \sum b'c'd'(\beta-\theta)(\gamma-\theta)(\delta-\theta)/a'b'c'd'(\alpha-\theta)(\beta-\theta)(\gamma-\theta)(\delta-\theta) \equiv f(\theta)/a'b'c'd'(\alpha-\theta)(\beta-\theta)(\gamma-\theta)(\delta-\theta) \lesseqgtr 0$. The inequalities are to be reversed when three of the coefficients are negative. The roots of $f(\theta)=0$ give the paraboloids of the system. It is supposed that two of the paraboloids are elliptic.

5. There are several cases to consider. Let first $\alpha>\beta>\gamma>\delta$ when the quartic curve is imaginary. The signs of $F(\lambda)$ or $f(\theta)$ for $\alpha, \beta, \gamma, \delta$ are $- + + -$. Hence λ_1, θ_1 lies in the internal (α, β), λ_2, θ_2 in (γ, δ) λ_3, θ_3 may lie either in (∞, α) or $(\delta, -\infty)$. The sign of $\sum{}^1/a'(\alpha-\lambda_1)$ is the same as that of $(\sum 1/a')(\lambda_1-\theta_1)(\lambda_1-\theta_2)(\lambda_1-\theta_3)$ i.e. that of $(\lambda_1-\theta_1)(\lambda_1-\theta_2)$ because $(\sum 1/a')(\lambda_1-\theta_3)>0$. Similarly the sign of $\sum l^2/a'(\alpha-\theta_1)$ is that of $(\theta_1-\lambda_1)(\theta_1-\lambda_2)$. Similar conclusions hold for λ_2, θ_2. Hence in this case the two conicoids are both hyperboloids of two sheets or one ellipsoid, one hyperboloid or both ellipsoids according as the plane cuts neither, one or both the paraboloids.

6. Secondly let $\alpha>\gamma>\beta>\delta$. The signs in this case are $-, -, -, -$. Hence λ_1, λ_2 and θ_1, θ_2 lie in (α, γ) or (β, δ). As before the sign of $\sum 1/a'(\alpha-\lambda)$ is that of $(\lambda-\theta_1)(\lambda-\theta_2)$ for λ_1, λ_2 and that of $\sum l^2/a'(\alpha-\theta)$ is that of $(\theta-\lambda_1)(\theta-\lambda_2)$ for either θ_1 or θ_2. Hence in this case the two conicoids are both two sheeted hyperboloids when the plane does not cut either paraboloid, one ellipsoid and one hyperboloid when the plane cuts only one paraboloid. But if the plane cuts both no such general conclusion can be drawn.

Lastly let $\alpha > \gamma > \delta > \beta$. The signs of $F(\lambda)$ and $f(\theta)$ are now $-$, $-$, $+$, $+$. Hence as only one conicoid is a hyperboloid of one sheet and one hyperbolic paraboloid $\sum l^2/a' > 0$ and $\sum '/a' > 0$. The roots λ_1 λ_2 and θ_1, θ_2 lie either in (α, γ) or (δ, β), λ_3, θ_3 in (γ, δ). Hence remembering that when three of the coefficients are negative (as is so when λ_1, λ_2; θ_1, θ_2 lie in (δ, β)) the inequalities are to be reversed, *we have*, λ_1, λ_2 *determines* a two-sheeted hyperboloid or an ellipsoid if $(\lambda - \theta_1)(\lambda - \theta_2) \lesseqgtr 0$ for λ_1, λ_2 and the plane cuts or not the paraboloids if for θ_1 or θ_2, $(\theta - \lambda_1)(\theta - \lambda_2) \lesseqgtr 0$. Hence the conclusions arrived at are the same as above. The quartic curve in either case is real.

7. The results therefore may be summed up as follows :—

"*If the plane meets the quartic curve in four real points, the three conicoids are all hyperboloids of one sheet; if in two real points only one conicoid is real and is a hyperboloid of one sheet; if it cuts it in imaginary points and if only one or all the three paraboloids are real and hyperbolic, one conicoid is a hyperboloid of one sheet, the other two of two sheets. If the points of section of the plane and the curve are imaginary and two of the paraboloids are elliptic, one conicoid is always a one-sheeted hyperboloid, the other two both hyperboloids of two sheets if the plane does not cut either paraboloid, one a two-sheeted hyperboloid and the other an ellipsoid if it cuts only one paraboloid. The nature of the two conicoids is left in doubt when the plane cuts both the paraboloids except when the quartic curve is imaginary when both are ellipsoids.*"

I gratefully acknowledge certain suggestions from Dr. Vaidyanathaswamy in preparing this paper.

A CRITERION FOR THE SOLUBILITY BY RADICALS OF THE GENERAL QUINTIC

BY

YASHPAULRAJ BHALOTRA, *Lahore*

1. The classical test (as developed in Cajori's theory of equations) for the solubility by radicals of the equation $x^5+cx+d=0$ is that if it be irreducible, it is metacyclic if and only if the equation

$$(z^3-5cz^2+15c^2z+5c^3)^2=(4^4c^5+5^5d^4)z$$

has a rational root.

Now, when the general quintic is reduced to the form $x^5+cx+d=0$, the values of c and d become in general very large. So it is worth while to have a test applicable to the quintics of the form

$$x^5+ax^3+bx^2+cx+d=0.$$

2. THEOREM :—*The equation (supposed irreducible)*

$$x^5+ax^3+bx^2+cx+d=0 \qquad \text{(I)}$$

is soluble by radicals if and only if the equation

$$\{z^3-(20c+3a^2)z^2+(240c^2-400bd-8a^2c+16ab^2+3a^4)z+(320c^3-1600bcd$$
$$-64b^4+4000ad^2+224ab^2c-176a^2c^2-80a^2bd-16a^3b^2+28a^4c-a^6)\}^2=2^{10}\mathrm{D}\cdot z$$

where D is the discriminant of (I), has a rational root.

Proof :—

(The proof in essence is on the same lines as given for the classical test in Cajori's *theory of equations*).

Let x_0, x_1, x_2, x_3, x_4 be the roots of (1)

We consider the expression

$$y\equiv(x_0x_1+x_1x_2+x_2x_3+x_3x_4+x_4x_0)-(x_0x_2+x_1x_3+x_2x_4+x_3x_0+x_4x_1)$$

operated upon by the Symmetric group. y yields 12 values of which 6 differ from the other 6 only in sign. Let one set of 6 values be $y_1, y_2, \cdots y_6$. Also let the equation of which they are the roots be

$$y^6+a_1y^5+a_2y^4+a_3y^3+a_4y^2+a_5y+a_6=0.$$

Its coefficient $a_1, a_2, \cdots a_6$ are not necessarily rational numbers but they are symmetric functions of $y_1, y_2, \cdots y_6$. Consider $y_1, \cdots y_6$

as the functions of $x_0, x_1, \cdots x_4$, and operate upon them with the alternating group; the values $y_1, \cdots\cdots y_6$ are merely permuted amongst themselves. Substitutions which do not belong to the alternating group bring about a change in sign. The coefficients $a_1, \cdots a_6$ are therefore either symmetric or alternating functions of $x_0, x_1, \cdots x_4$; of these a_2, a_4, a_6 are symmetric functions because, being homogeneous function of even degree, they are not affected by changes of sign in $y_1, \cdots y_6$; on the other hand, a_1, a_3, a_5 are alternating functions of $x_0, x_1, \cdots x_4$, being homogeneous functions of odd degree.

If D is the discriminant of the quintic, then $\sqrt{D}$ is a function of $x_0, \cdots x_4$ belonging to the alternating group. Hence the coefficients a_1, a_3, a_5 are of the form $m_1\sqrt{D}$, $m_2\sqrt{D}$, $m_3\sqrt{D}$ where m_1, m_2, m_3 are symmetric integral functions. With respect to $x_0, \cdots x_4$ it is seen that a_1, is of the second degree. But a_1 is also of the form $m_1\sqrt{D}$ where m_1 is integral and $\sqrt{D}$ is of the tenth degree. Hence, we have $m_1=0$. Similarly a_3 being the 6th degree we have $m_3=3$. On the other hand a_5 and $\sqrt{D}$ are both of the 10th degree.

Write $a_5=m\sqrt{D}$. Then (1) becomes

$$y^6+a_2y^4+a_4y^2+m\sqrt{D}\,y+a_6=0 \qquad \text{(II)}$$

In (1) a, b, c, d are of the degrees 2, 3, 4, 5 respectively. Since a_2, a_4, a_6 are of degrees 4, 8 and 12, we may write

$$a_2=m_1c+m_2a^2$$

$$a_4=m_3c^2+m_4bd+m_5a^2c+m_{16}ab^2+m_6a^4.$$

$$a_6=m_7c^3+m_8bcd+m_9b^4+m_{17}ad^2+m_{10}a^2bc+m_{11}a^2c^2+m_{12}a^2bd+m_{13}a^3b^2 + m_{14}a^4c+m_{15}a^6.$$

Put $c=-1$, $a=b=d=0$; Then $D=-4^4$ and (II) becomes

$$0=y^6-m_1y^4+m_3y^2+16imy-m_7=y^6+20y^4+240y^2+512iy-320;$$

that is $m_1=-20$, $m_3=240$, $m=32$, $m_7=320$

Put $b=-1$, $a=c=d=0$, we get

$$y^6+m_9=y^6-64.$$

$$\therefore\quad m_9=-64.$$

The above calculation illustrates the method of evaluating the quantities $m_1, m_2\cdots, m_{17}$. The following succession of special cases apart from the two already used and detailed calculations lead to the value of these seventeen quantities:—

(i)	$a=0$,	$b=-10$,	$c=15$,	$d=-6$
(ii)	$a=-1$,	$b=c=d=0$		
(iii)	$a=-2$,	$b=0$,	$c=1$,	$d=0$
(iv)	$a=-3$,	$b=2$,	$c=d=0$	
(v)	$a=-6$,	$b=8$,	$c=-3$,	$d=0$
(vi)	$a=-10$,	$b=20$,	$c=-15$,	$d=4$
(vii)	$a=-10/3$,	$b=0$,	$c=5$,	$d=-8/3$
(ix)	$a=1$,	$b=0$,	$c=1$,	$d=0$

The final result is

$$\{y^6-(20c+3a^2)y^4+(240c^2-400bd-8a^2c+16ab^2+3a^4)y^2$$
$$+320c^3-1600bcd-64b^4+400ad^2+224ab^2c-176a^2c^2-80a^2bd^2-16a^3b^2$$
$$+28a^4c-a^6\}^2=2^{10}Dy^2.$$

Putting $y^2=z$ we get the required result.

3. As an application of this theorem considered Ramanujan's quintic
$$x^5-x^4+x^3-2x^2+3x-1=0$$

Put $x^2=\dfrac{1}{X+1}$. Then the above quintic becomes
$$X^5-10X^3-23X^2-22X-9=0$$

In this case the equation of the theorem is
$$(z^3+140z^2-3680z-72064)^2=2^{10}\cdot 79^2\cdot 9^2z^2,$$
which has a rational root 16.

Hence $x^5-x^4+x^3-2x^2+3x-1=0$ is soluble by radicals.

4. Putting $a=0$, we get a test for equations of the form $x^5+bx^2+cx+d=0$. Dr. S. Chowla developed the test for such equations independently and has solved Ramanujan's second quintic, viz. $x^5+2x^4+2x^3+x^2-1=0$.

By putting $x=\sqrt{y}$ it becomes $y^5+3y^3+2y-1=0$. Here $b=3$, $c=2$, $d=-1$ and the z-equation becomes
$$(z^3-40^2+2160\ z+64\cdot 109)^2=7520^2\cdot z,$$
or putting $z=4u$, $(u^3-10u^2+135u+109)^2=235^2u$,

which obviously has a root $u=1$.

Examples: Prove the solubility by radicals of

(1) $x^5-x^4-3x^3-3x^2-2x-1=0$;

(2) $x^5-3x^4-x^3+2x^2+x-1=0$.

NOTES AND DISCUSSIONS

The Bear-Clock

In a speech made on the occasion of inaugurating the Mathematical Association of Maharaja's College, Mysore, Principal J. C. Rollo * cited the formula:

$$T = K - 2(M + H) \quad \ldots \quad (1)$$

(where T is the local time, M the date in months, and H the time indicated by the Bear-Clock, to be defined presently, and K is a suitable constant), as one communicated to him by a friend in the astronomical atmosphere of Kodaikanal, and of which he was able to supply a proof himself.

Imagine a huge clock-dial in the sky with the centre at the pole-star and the line from it to the Pointers of the Great Bear (α, β, Ursæ Majoris) as the single and unique hour-hand indicating a kind of sidereal time, which we may call the Bear-time. This imaginary Bear-Clock may be further supposed to be graduated in the same way as the ordinary clock, though the indicator actually moves counter-clockwise. For an accurate reading of the clock, the reader should practise reading time from ordinary clock, first without the help of the minute-hand and later even without the help of the numbers on the dial.

We now proceed to give briefly the orthodox proof of the formula (1).

We know that

$$\text{'Hour-angle of a Star'} + \text{its 'Right-Ascension'} = \text{Sidereal Time}$$

Remembering that the R. A. of Dubhe (α, Ursæ Majoris) is 10·98 hrs., we get

$$2(12 - H) + 10{\cdot}98 = T + \text{R. A. of } \odot \text{ in hours}$$

$$= T + \frac{2}{30}(\text{R. A. in degrees}). \qquad (2)$$

---

* It is axiomatic that the formula should make a wide appeal, one evidence being that it has excited the dormant mathematical interests of Mr. J. C. Rollo, a Professor of English for several years.

Assuming the Sun's motion in R. A. to be uniform, we write

R. A. of the Sun at a given date (30 M days after the mid-night of the 31st of the previous December)

$$= \{(30\text{ M} - 82)\ 360/365{\cdot}25\}^\circ$$

$$= (30\text{M} - \frac{3\text{ M}}{7} - 80{\cdot}8)^\circ$$

Substituting this value in (2), we obtain

$$\text{T} = \frac{121}{3} - 2(\text{M}+\text{H}) + \frac{\text{M}}{35} \qquad (3)$$

an approximate formula, slightly more refined than (1).

Expressing M in terms of the date as usually given, viz. dth day of the mth month, we may modify (3) as

$$\text{T} = 2\{21 \text{ hrs. } 10 \text{ min.} - m \text{ hrs. } 2d \text{ min.} - \text{H}\} + c \text{ min.} \qquad (4)$$

where c is a correction in minutes which varies with calendar months according to the following scheme:

Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
2	−1	9	7	9	6	8	6	3	5	3	5

The correction takes into account the factor $\frac{\text{M}}{35}$ as well as the inequalities in the lengths of the calendar months.

The interested reader may verify the results given by our formula (4) with readings from any Planisphere, say Philip's, showing the principal stars visible for every hour in the year.

Mysore. A. A. KRISHNASWAMI AYYANGAR.

On the Definition of Oscillation

If $\phi(n)$ does not tend to a limit, nor to $+\infty$, nor to $-\infty$, then it is said to oscillate.

So far as I know, this is the only definition given for oscillation. This definition is defective in that it is purely a negative one. But it is possible to give a positive definition as follows:—

Definition: If there are two constants h_1 and h_2 such that $\phi(n_1) \leqslant h_1 < h_2 \leqslant \phi(n_2)$ for infinitely many values of n_1 and n_2, then $\phi(n)$ oscillates.

Hereafter when we say that $\phi(n)$ oscillates, we consider only the new definition but not the usual definition given.

Suppose that $\phi(n)$ does not tend to a limit nor to $+\infty$, nor to $-\infty$. Then should $\phi(n)$ necessarily oscillate or can it behave in any other way not covered by any of the four cases?

First of all, let us suppose that $\phi(n)$ can take a value larger than any arbitrarily large number and at the same time $\phi(n)$ does not tend to $+\infty$. In this case, if there is no number k such that $\phi(n_1)<k$ for infinitely many values of n_1, $\phi(n)$ will tend to $+\infty$. So there is a number k such that $\phi(n_1)<k$ for infinitely many values of n_1. From the given condition we get that $\phi(n_2)>k+1$ for infinitely many values of n_2. So we can put $h_1=k$ and $h_2=k+1$ in the definition and consequently $\phi(n)$ oscillates. Similarly we can prove that, if $\phi(n)$ takes values less than any arbitrary negative number and at the same time $\phi(n)$ does not tend to $-\infty$, then $\phi(n)$ oscillates. By combining these two, we arrive at

THEOREM A: *An unbounded sequence which does not tend to* $+\infty$, *nor to* $-\infty$, *necessarily oscillates.*

Then we have to consider a bounded sequence. Should a bounded sequence which does not tend to a limit necessarily oscillate, or can it behave in any other way? Is it possible to prove that there are only two ways for the behaviour of a bounded sequence, or should we assume this as an axiom? We have to consider the following:

STATEMENT: *A bounded sequence either tends to a limit or oscillates.*

In proving the general principle of convergence, the difficult part is the proof of the sufficiency condition. If we assume the above statement as an axiom, the sufficiency condition is a very easy and immediate corollary from it; because, from the definition, a sequence $\phi(n)$ oscillates only if there is a positive number $h>0$ such that $\phi(n_2)-\phi(n_1)\geqslant h$ $(h=h_2-h_1)$ for infinitely many values of n_1 and n_2; and this is contrary to the condition of the general principle. So if we assume that the above statement is true, the sufficiency condition does not require any proof.

But there is a better mode of approach; we can prove the above statement; as a matter of fact, it is an immediate consequence of the general principle of convergence,

THEOREM B: *A bounded sequence either tends to a limit or oscillates.*

Given any sequence $\phi(n)$, either there is a constant h such that $\phi(n_2) - \phi(n_1) > h$ for infinitely many values of n_1 and n_2, or there is no such constant.

The non-existence of such a constant means that corresponding to every ε, there is a n_0 such that $|\phi(n_2) - \phi(n_1)| < \varepsilon$ for all $n_2 > n_1 \geqslant n_0$. Therefore by the general principle $\phi(n)$ tends to a limit.

Let us consider the case when h exists. Let M be an upper bound and m a lower bound of the sequence, and $k = [2(M-m)/h] + 1$. Divide the interval into k equal parts. Since n_1 is an infinite set there should be at least one sub-interval, say the $(r-1)$ th, which contains an infinity of members of the sequence $\phi(n_1)$. Since the length of a sub-interval $< \frac{1}{2}h$ and $\phi(n_2) - \phi(n_1) > h$, for the corresponding values of n_2, $\phi(n_2)$ should lie either in the $(r+1)$th or any succeeding interval. Let h_1 be the left hand end point and h_2 the right hand end point of the rth interval. Then for this infinite set of values of n_1 and n_2,

$$\phi(n_1) \leqslant h_1 < h_2 \leqslant \phi(n_2).$$

So $\phi(n)$ oscillates and theorem B is completely proved.

By combining theorems A and B, we get the

THEOREM: *A sequence $\phi(n)$ should either tend to $+\infty$, or to $-\infty$, or to a finite limit, or oscillate.*

Travancore University. S. S. PILLAI,

Lecture Notes

1. *Uniform convergence; A correction*

Mr. C. Dockeray, in his *Elementary Treatise on Pure Mathematics* (1934) (vide p. 462), says that the series

$$\sum_{n=1}^{\infty} x^3 \left(\frac{1-x^2}{1+x^2}\right)^{n-1}$$

is continuous when $x=0$. having as its sum $\frac{1}{2}x(1+x^2)$, *although it is not uniformly convergent in an interval containing the point* $x=0$.

The statement is not correct; the *series is uniformly convergent for all values of x*. Since when x is negative the series merely changes sign, we will prove that the series is uniformly convergent in the interval $0 \leqslant x \leqslant 1$. That is, we have to show that corresponding to any given ε, a number m can be found such that

$$\sum_{n=m+1}^{\infty} x^3 \left(\frac{1-x^2}{1+x^2}\right)^{n-1} < \varepsilon$$

for all values of x in the interval, *m being independent of x.*

Given ε, splitting the interval (0, 1) into the two intervals $(0, \varepsilon/_2)$, $(\varepsilon/_2, 1)$, for any value of x in the interval $(\varepsilon/_2, 1)$

$$\sum_{n=m+1}^{\infty} x^3 \left(\frac{1-x^2}{1+x^2}\right)^{n-1}$$

$$= x^3 \left(\frac{1-x^2}{1+x^2}\right)^m \frac{1+x^2}{2x^2}$$

$$= \tfrac{1}{2}\, x\, (1+x^2) \left(\frac{1-x^2}{1+x^2}\right)^m$$

$$\leqslant \tfrac{1}{2}\, 1\cdot(1+1) \left(\frac{1-\varepsilon^2/_4}{1+\varepsilon^2/_4}\right)^m$$

$$< \frac{1}{(1+\varepsilon^2/_4)^m}$$

$$< \frac{1}{1+m\,\varepsilon^2/_4}$$

$$< 4/m\,\varepsilon^2$$

$$< \varepsilon \text{ (if } m > 4/\varepsilon^3)$$

Now, for this value of m_1,

$$\sum_{m+1}^{\infty} x^3 \left(\frac{1-x^2}{1+x^2}\right)^{n-1} < \varepsilon$$

even if x lies in the interval $(0, \varepsilon/2)$

For,

$$\sum_{m+1}^{\infty} x^3 \left(\frac{1-x^2}{1+x^2}\right)^{n-1}$$

$$= \tfrac{1}{2}x\,(x^2+1) \left(\frac{1-x^2}{1+x^2}\right)^m$$

$$< \tfrac{1}{2}x \cdot (1+1) \cdot 1 < x < \varepsilon.$$

Hence, the 'm' found, viz $m = \mathrm{I}\left(\frac{4}{\varepsilon^3}\right) + 1.$

is *independent of* x and is such that the remainder after 'm' terms

is less than ε, for all the values of x in the interval (0, 1). Hence the series is uniformly convergent in (0, 1).

St. Joseph's College
Trichinopoly.

S. SURYANARAYANA IYER

[*Note*: That the series is uniformly convergent in $0 \leqslant x \leqslant 1$ can also be seen from the fact that the sum is continuous at $x=0$ and the terms are non-negative. A general theorem due to Dini now gives the result. *Ed.*]

2. *Criteria for an ellipse*

It is given in text books that $C = ab - h^2 > 0$ is the condition that the equation.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

represents an ellipse. But we are not so much interested in knowing whether (1) gives an ellipse as in knowing whether it gives a real ellipse. And for this the above condition is only necessary not sufficient. The necessary and sufficient condition is that the roots of the quadratic

$$\lambda^2 - (a+b)\lambda + (ab - h^2) = 0 \tag{2}$$

have the sign opposite to that of the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

To prove this, we have only to refer (1) to its centre. With axes parallel to the original ones the equation (1) becomes

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{C} = 0.$$

If now we rotate the axes so as to coincide with the principal axes, the new equation may be written as

$$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{C} = 0 \tag{3}$$

where, using invariants,

$$\lambda_1 + \lambda_2 = a + b, \ \lambda_1\lambda_2 = ab - h^2;$$

and these equations show at once that λ_1 and λ_2 are the roots of (2).

The conditions for real ellipse are (i) λ_1 and λ_2 must have the same sign, (ii) this sign must be the opposite of that of $\frac{\Delta}{C}$. From (i), $\lambda_1\lambda_2 = C > 0$, and hence from (ii) we get the necessary and sufficient conditions as stated.

As an illustration, we consider the conic given by

$$5x^2 + 6xy + 5y^2 + 4x - 2y + 3 = 0.$$

Here $C = 5 \cdot 5 - 3^2 > 0$, but the equation gives an imaginary ellipse, for $\lambda_1 = 2$, $\lambda_2 = 8$ and $\Delta = 21$.

University of Travancore. V. BALACHANDRAN.

3. *The greatest angle of a triangle*

Given the equations of the sides of a triangle, we can know only the absolute values of the tangents of the angles. How can we determine the angles of the triangle without any ambiguity? I shall give here some trivial but very useful results:

If A be the greatest angle of a triangle $|\tan A|$ is greater than $|\tan B|$ and $|\tan C|$; again if B, C be the two lesser angles of any triangle, then $\tan B \times \tan C$ is greater than, equal to, or less than 1 according as the triangle has its greatest angle acute, right, or obtuse, and conversely. Hence, if $|\tan A|$, $|\tan B|$, $|\tan C|$ be known, we have only to examine the product of the two lesser ones to determine the species of the triangle.

In an acute-angled triangle, the product of the tangents of the two lesser angles lies between 1 and 3.

To prove these results, you require not more than one second of thought! Just try please.

Mysore, A. A. KRISHNASWAMY AYYANGAR.

4. *F-Conic as a rectangular hyperbola*

The F-conic of a conic Σ and a parabola p such that the centre O of the conic Σ lies on the directrix of p is a rectangular hyperbola.*

Let P and P′ be the intersections of the F-conic with the line at infinity. Now the tangents from P to Σ are conjugate for *p*, *i.e.* the tangent from P to the parabola passes through O, similarly

* This conic is known as the harmonic-locus of two conics. G. Salmon—*Conic Sections.*

that from P′ also passes through O. Now OP, OP′ being tangents to the parabola from a point O on its directrix must be at right angles, but they are lines parallel to the asymptotic directions of the F-conic. Hence it is a rectangular hyperbola.

SAHIB RAM MANDAN.

5. *Inconic as a parabola*

An inconic, i.e. a conic inscribed to a triangle will be a parabola if the Hessian line passes through the centroid of the triangle.*

Now an inconic may be considered as the envelope of the lines polar to the points on the Hessian line w.r.t. the triangle. Hence it touches the line at infinity *i.e.* it will be a parabola if the pole of the line at infinity w.r.t. the triangle, *i.e.* the centroid lies on the Hessian line.

Lahore, SAHIB RAM MANDAN.

RECEIVED FOR REVIEW

F. W. LEVI: Finite Geometrical systems (Six public lectures delivered in February 1940) Calcutta University, 1942 pp 51.

REVIEWS

W. L. FERRAR M. A., F. R. S. E. *Algebra, A Textbook of Determinants, Matrices, and Algebraic Forms* pages 199. Oxford University Press Price 12 sh. 6 d.

As the author explains in the introduction, the book has been written with a view of putting, within the reach of undergraduates, an elementary course in matrices and quadratic forms. The book has been written with the care and selection requisite for this purpose. The examples too have been carefully chosen, and are sometimes provided with hints. We are sure that the book supplies a real need, and will be found very useful.

The reviewer however feels strongly that, at the present stage of mathematical education, this is not exactly the sort of book which should be written for undergraduates. It is quite true that there is no book which gives the elementary notions regarding matrices which are so necessary in applications. But modern algebra has taken tremendous strides in this century, and it is a no

less urgent need to write books through which the modern outlook and ideas may filter down into the College course. In fact, a college text-book on matrices written today, ought *not* to be the same in outlook and method, as a book written in the 19th century. The author himself must have felt some qualms of conscience on this matter, and has put in a preliminary note defining number-rings and fields, where he remarks that 'most of the propositions of the book apply to *any* field of numbers'. If this is so, then, without enlarging the book or raising its standard, the propositions of the book could have been easily stated for abstract fields, and particularised, as instances, for the elementary fields viz, complex numbers, real numbers, rational numbers, and residue-classes modulo a prime. If this had been done, the book would have achieved the introduction of the field-idea into undergraduate work.

But as a matter of fact the preliminary note seems to have been an after-thought, and stands in no close relation with the methods of the book. I believe that as the book stands, it should be replaced sooner or later by a book of roughly the same scope, so written as to introduce effectively the field-idea.

R. VAIDYANATHASWAMY.

M. S. GODBOLE and R. P. SHINTRE. *Elementary Plane Trigonometry. Aryabhushan Press, Poona.*

The purpose of the book seems to be to give an elementary knowledge of the subject up to about the Intermediate standard of the South Indian Universities. There is a short historical introduction and appropriate historical notes in the course of the book.

Starting with the definitions of the Trigonometric ratios, the beginner is enabled to solve elementary problems involving right-angled triangles. The generalised notion of angles and the variations in the trigonometric ratios of these angles are then dealt with, and illustrated graphically. The limits of cot θ and cosec θ as $\theta \to 0$ and those of tan θ and sec θ as $\theta \to \pi/2$ are explained clearly with the help of diagrams. Then follow chapters relating to the simpler properties of triangles, compound angles are taken up next, leading to the treatment of multiple angles and summation of elementary series. The book concludes with the solution of problems on Heights and Distances and geometrical properties of the triangle and the quadrilateral.

The authors have taken great pains to get a good collection of examples both useful from the students' point of view and illustrative of the varied applications of the subject.

A bigger type for the illustrative examples and for the exercises a bolder type for the principal results and a collection of the formulae for ready reference would render the book more attractive and useful.

S. A. M.

T. K. RAGHAVACHARI, M.A. *An Introduction to Analytical Geometry and Calculus.* Oxford University Press, 1941. Pp. 192.

This book covers the syllabuses for the Intermediate Examination of the South Indian Universities. It commences with an excellent and interesting historical introduction covering about eight pages. The author has devoted a good deal of thought and has shown considerable originality in the presentation of the subject. One noteworthy feature of the book is the use of the idea of projection of segments of lines in the proofs of several results in Analytical Geometry.

The author has succeeded in giving a clear and rigorous exposition of the idea of a limit so fundamental in Calculus and a good treatment of "limits" and "continuity." It is possible that a student of the Intermediate standard may not be able to appreciate the full significance of the points discussed; nevertheless the author deserves praise for giving a lucid exposition of a topic usually felt difficult even by students of higher classes.

The book contains many worked examples and numerous exercises. The printing is neat and the get-up of the book is good.

V. G.

GLEANINGS

He told friends in later years how Büttner (his teacher) gave the class a long, difficult exercise in calculation; he noticed the law of the arithmetical series. His fellow pupils were trying to find the solution by tedious addition. Suddenly he threw the slate, containing the answer, on the teacher's desk with the overweening exclamations. "There 'tis" (Dar licht Se! or Ligget Se!) We are told that this accomplishment made such an impression on Büttner that Gauss was allowed to carry out into the yard at home his spinning wheel at which he had been compelled to spin his portion of flax, and he was allowed to chop it up with an axe for kindling.

Gauss never liked to teach, because it robbed him of time for research. According to his notions, the ideal plan would be to lecture on research in which one might be enga[illegible]t the moment or in which one might be specially interested. Neverthel[illegible]now that Gauss was a great teacher in his later years, that his lectures [illegible]ful, clear presentations of the topic.

From G. W. DUNNINGTON'S
Carl Friedrich Gauss.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been elected members of the Indian Mathematical Society:

1. M. S. Srinivasan Esq., No. 3, Chidambarier Stores, Santhanagopalapuram, Villupuram.

2. P. B. Bhattacharya Esq., M.Sc., St. Stephen's College, Delhi.

3. Faqir Chand Auluck, M.A., Lecturer, Dayal Singh College, Lahore.

4. Sahib Ram Esq., M.A., Lecturer, Punjab University, Lahore.

5. Miss Veda Thakurdas, M.A. (Cantab) M.A. (Punjab), Principal, Islamia College for women, Lahore.

6. D. J. Madan Esq., M.Sc., LL. B., Lecturer, D. J. Sind College, Karachi.

7. A. C. Chaudhury Esq., Dept. of Mathematics, Calcutta University, 21/1A Shyamananda Road, P. O, Kalighat, Calcutta.

8. B. C. Chatterjee Esq., M.Sc., Mathematics Department, Calcutta University, 2/16B, Kanxila Road, Ballygunj, Calcutta.

9. Dr. V. Ganapathi Iyer M.A., D.Sc., Lecturer in Mathematics, Annamalai University, Annamalainagar.

10. Gobind Ram Seth Esq., M.A., Hindu College, Delhi.

The Indian Mathematical Society celebrated its 12th Conference at Aligarh on the 27th, 28th and 30th of December 1941. A detailed report of the Conference will be published in the next issue of *The Mathematics Student.*

As already announced in the June 1940 issue of the *Student*, Dr. Hansraj Gupta is collecting details regarding the life and work of mathematicians engaged in research in India with a view to their publication. Such a work, he points out, will help to bring together workers in similar fields and to serve as an inspiration to younger men. He appeals to all mathematicians in India for cooperation and requests them to send the following details about themselves before the end of June 1942:

(1) Name in full (2) Present Address (3) Branches of mathematics with which their research deals (4) Date, place and province of birth (5) Institutions where educated, with dates, degrees obtained and special distinctions (6) Names of learned Societies of which they are members and offices held in such Societies (7) Complete lists of published papers with full details regarding volume, year, pages etc.; reprints to be sent if available (8) Names of books published in Higher Mathematics with [illegible] details, and (9) Any other points of interest relating to themselves.

Mr. Abdullah Butt has filled the post [illegible]ecturer in the Aligarh University vacated by the resignation of Dr. [illegible]n, and Mr. Omar Ali Siddiqi has been appointed to the post created b[illegible] motion of Mr. Butt.

The title of Khan Bahadur has been co[illegible] on Mr. Abdul Majid Kureishy, Head of the Mathematics Department, [illegible]m University, Aligarh.